PARTIAL DIFFERENTIAL EQUATIONS

XAVIER ROS OTON

3. Nonlinear elliptic PDE and the Calculus of Variations

(1) Let $u \in C^\infty(\overline{\Omega})$ be a local minimizer of the functional

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u)$$

with fixed boundary condition $u = g$ on $\partial\Omega$, that is, there exists $\delta > 0$ for which

$$\mathcal{E}(u+\xi) \geq \mathcal{E}(u) \qquad \forall \xi \in C_c^\infty(\Omega) \quad \text{with} \quad \|\xi\|_{L^\infty(\Omega)} \leq \delta.$$

Compute a "second derivative" of the functional to deduce that

$$\int_\Omega |\nabla \eta|^2 \geq \int_\Omega f'(u)\eta^2 \quad \text{for every} \quad \eta \in C_c^\infty(\Omega),$$

where $F' = f$.

Note: This inequality is satisfied by local minimizers of the functional. However, it does not hold in general for all solutions of the corresponding PDE.

(3 points)

(2) Given $\varphi \in C_c^\infty(\Omega)$, prove that there exists a function u that minimizes the Dirichlet integral among all functions $w \in H_0^1(\Omega)$ satisfying $w \geq \varphi$ in Ω.

(2 points)

(3) Let $n \geq 3$. Prove that $u(x) = \log \frac{1}{|x|^2}$ belongs to $H^1(B_1)$ and is a solution of

$$\begin{cases} -\Delta u = \kappa_n e^u & \text{in} \quad B_1 \\ \quad u = 0 & \text{on} \quad \partial B_1, \end{cases}$$

for some constant $\kappa_n > 0$.

This shows that solutions to nonlinear PDE can be singular, i.e., with $u \to \infty$ at an interior point.

(2 points)

(4) Find a positive solution of

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \quad B_1$$

of the form $u(x) = a/(1-|x|^2)^\beta$ for positive constants a, β.

This shows that solutions to nonlinear PDE can be smooth inside a domain and yet become infinity *everywhere* on its boundary.

(2 points)

(5) (i) Prove that for any $\alpha \in (0,1)$ there exists a minimizer $u \in H^1_0(\Omega)$ of the functional

$$\mathcal{E}(w) := \int_\Omega \tfrac{1}{2}|\nabla w|^2$$

over all functions $w \in H^1_0(\Omega)$ satisfying

$$\int_\Omega |w|^{\alpha+1} dx = 1.$$

Moreover, use that $|\nabla|u|| = |\nabla u|$ to prove that we can take $u \geq 0$.

(ii) Prove that such minimizer $u \in H^1_0(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta u &= \lambda u^\alpha & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

for some constant λ.

(4 points)

(6) Given a bounded smooth domain $\Omega \subset \mathbb{R}^n$ and smooth map $\vec{g} : \partial\Omega \to \mathbb{S}^m$, prove that there exists a map $\vec{u} : \Omega \to \mathbb{S}^m$ that minimizes the Dirichlet energy $\int_\Omega |\nabla \vec{w}|^2 dx$ among all maps satisfying $\vec{w} = \vec{g}$ on $\partial\Omega$.

(4 points)

(7) Prove that, in dimension 1, there is no function $u \in C^2([-1,1])$ that minimizes the functional

$$\int_{-1}^{1} x^2 |u'(x)|^2 dx$$

among all C^2 functions satisfying $u(-1) = -1$ and $u(1) = 1$.

(3 points)

(8) Let $L(p,u,x)$ be smooth and uniformly convex in p, and let $u \in C^2(\Omega)$ be a minimizer of

$$\int_\Omega L(\nabla u, u, x)$$

among all functions satisfying $u = g$ on $\partial\Omega$.

Find the PDE satisfied by u inside Ω.

(3 points)

(9) Let $\Gamma = \{(x,y) \in \Omega \times \mathbb{R} : y = u(x)\} \subset \mathbb{R}^{n+1}$ be a smooth hypersurface, given as the graph of a function $u \in C^\infty(\Omega)$. Prove that the mean curvature of Γ at a point $x \in \Omega$ is given by

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)$$

(3 points)

(10) Assume that $u \in C^\infty(\overline{\Omega})$ is a minimizer of the area functional

$$\int_\Omega \sqrt{1+|\nabla w|^2}\,dx,$$

among all functions with fixed boundary conditions $w = g$ on $\partial\Omega$, and with the fixed constraint

$$\int_\Omega w = c_0.$$

Prove that the graph of u is a hypersurface of constant mean curvature.

Deduce that if $U \subset \mathbb{R}^n$ is a bounded smooth set that minimizes the surface area among all sets with volume 1, then ∂U has constant mean curvature.

Hint: Use that the mean curvature is given by $H := \operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2})$

(4 points)

(11) Given $\phi \in C^\infty(\overline{\Omega})$, find a functional $\int_\Omega L(\nabla u, x)$ so that its corresponding Euler-Lagrange equation is the PDE

$$-\Delta u + \nabla\phi \cdot \nabla u = 0 \quad \text{in} \quad \Omega.$$

Hint: Try functionals with an exponential term.

(3 points)